Axiomatizing Modal Logic over Semilattices

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Newborn interest in studying modal logics over algebraic structures has emerged in recent years, with notable work on groups by [2] and on lattices by [4]. In this abstract, we contribute to this line of research by taking our class of frames to be (join-)semilattices and equipping our language with a binary modality '$\langle\mathrm{sup}\rangle$' for the join-operation. This yields a so-called modal information logic, proposed in [1] as a general logical base theory of information.

[3] proves that the modal information logics over preorders and posets coincide, are decidable and finitely axiomatizable. While it is straightforward to see that going one step further to semilattices makes for a different logic, developing a full axiomatization has proven challenging. In this talk, we present a solution to this problem by axiomatizing the logic through an infinite scheme. The proof is technical and lengthy, but we believe that it offers valuable additions to the toolbox of techniques for (modal) completeness proofs. Accordingly, our emphasis will be on accenting key ideas through an informal presentation.

Defining the logic

We continue by setting out our targeted logic.

Definition 1 (Language)**.** The language $\mathcal{L}_M$ is defined using a countable set of proposition letters **Prop** and a binary modality '$\langle\mathrm{sup}\rangle$'. The formulas $\varphi \in \mathcal{L}_M$ are then given by the following BNF-grammar:

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\mathrm{sup}\rangle\varphi\varphi,$$

where $p \in \mathbf{Prop}$ and $\bot$ is the falsum constant. $\dashv$

Definition 2 (Frames and models)**.** A (Kripke) *semilattice frame* is a pair $\mathbb{F} = (S, \leq)$, where S is a set and $\leq$ is a join-semilattice on S (i.e., reflexive, transitive and with all binary joins).

A (Kripke) *semilattice model* is a triple $\mathbb{M} = (S, \leq, V)$, where $(S, \leq)$ is a semilattice frame, and $V : \mathbf{Prop} \to \mathcal{P}(S)$ is a valuation on S. $\dashv$

Definition 3 (Semantics)**.** For any semilattice model $\mathbb{M} = (S, \leq, V)$ and state $s \in S$, *satisfaction* of a formula $\varphi \in \mathcal{L}_M$ at s in $\mathbb{M}$ (written $\mathbb{M}, s \Vdash \varphi$) is defined as follows:

$\mathbb{M}, s \nVdash \bot$,		
$\mathbb{M}, s \Vdash p$	**iff**	$s \in V(p)$,
$\mathbb{M}, s \Vdash \neg\varphi$	**iff**	$\mathbb{M}, s \nVdash \varphi$,
$\mathbb{M}, s \Vdash \varphi \vee \psi$	**iff**	$\mathbb{M}, s \Vdash \varphi$ *or* $\mathbb{M}, s \Vdash \psi$,
$\mathbb{M}, s \Vdash \langle\mathrm{sup}\rangle\varphi\psi$	**iff**	there exist $t, t' \in W$ such that $\mathbb{M}, t \Vdash \varphi$, $\mathbb{M}, t' \Vdash \psi$, and $s = \sup\{t, t'\}$.

Validity of a formula $\varphi \in \mathcal{L}_M$ in a frame $\mathbb{F}$ (written $\mathbb{F} \Vdash \varphi$) is defined as usual. $\dashv$

*This abstract is based on results from my MSc thesis [3], defended in August 2022, at the *ILLC* at the *University of Amsterdam*, supervised by Johan van Benthem and Nick Bezhanishvili. I would like to express my gratitude to both of them, as well as to the two anonymous referees for their valuable comments on this abstract.

Definition 4 (Logic)**.** The modal (information) logic over semilattices is denoted MIL_{Sem} and defined as

$$MIL_{Sem} := \{\varphi \in \mathcal{L}_M \mid (S, \leq) \Vdash \varphi, \text{ for all semilattice frames } (S, \leq)\}. \qquad \dashv$$

Remark 5. As mentioned, it is easily seen that our logic differs from the corresponding logic over preorders or posets. In particular, the associativity formula

$$\text{(As.)} \ \langle \sup \rangle (\langle \sup \rangle pq) r \leftrightarrow \langle \sup \rangle p (\langle \sup \rangle qr),$$

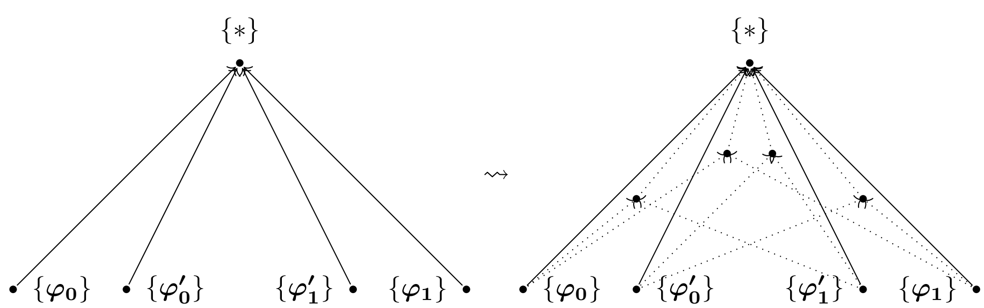
witnesses this by being valid on semilattices but not on posets. $\dashv$

Axiomatizing the logic: conceptual solution

Having introduced the logic, we proceed to present our solution to the axiomatization problem. Due to the complexity of the definitions involved, we refer the reader to Chapter 6 of [3] for details, and instead focus on conveying key ideas and heuristics.

Our approach is that of working out what axioms are needed to construct a semilattice model satisfying some maximal consistent set (MCS) $\Gamma_0 \supseteq X_0$ extending an arbitrary consistent set X_0.[1] Starting off, we define the single-state semilattice $(\{\{*\}\}, \{(\{*\}, \{*\})\})$ and 'label' it with our maximal consistent set of concern: $l(\{*\}) = \Gamma_0$. Constructing our model stepwise, the objective is then to prove a 'truth lemma'. If, say, $\{\langle \sup \rangle \varphi_0 \varphi_0', \langle \sup \rangle \varphi_1 \varphi_1'\} \subseteq \Gamma_0$, we would want to add corresponding points—for convenience called $\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi_1'}\}$—as in the left part of the figure below, and label them according to the existence lemma so that $\varphi_0 \in l(\{\boldsymbol{\varphi_0}\})$, etc.

The first complication then becomes that although, e.g., $\{*\} = \sup\{\{\boldsymbol{\varphi_0'}\}, \{\boldsymbol{\varphi_1'}\}\}$, we need not have $C_{\mathbf{Sem}} \Gamma_0 l(\{\boldsymbol{\varphi_0'}\}) l(\{\boldsymbol{\varphi_1'}\})$, where '$C_{\mathbf{Sem}}$' refers to the ternary relation of the canonical frame of the sought axiomatization. Therefore, we would want a formula $\pi_1 \in MIL_{Sem}$ somehow enabling us to add a point, $\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}$, and label it s.t. not only $C_{\mathbf{Sem}} l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\}) l(\{\boldsymbol{\varphi_0'}\}) l(\{\boldsymbol{\varphi_1'}\})$ but also $C_{\mathbf{Sem}} \Gamma_0 l(\boldsymbol{\varphi_0}) l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})$ and $C_{\mathbf{Sem}} \Gamma_0 l(\boldsymbol{\varphi_1}) l(\{\boldsymbol{\varphi_0'}, \boldsymbol{\varphi_1'}\})$. Taking this argument a step further, we would want π_1 to enable freely generating a semilattice modulo the requirements $\{*\} = \sup\{\boldsymbol{\varphi_0}, \boldsymbol{\varphi_0'}\}$ and $\{*\} = \sup\{\boldsymbol{\varphi_1}, \boldsymbol{\varphi_1'}\}$ (i.e., the RHS semilattice of the figure below) so that whenever $x = \sup\{y, z\}$, it is also the case that $C_{\mathbf{Sem}} l(x) l(y) l(z)$.



Now, it is obviously false that whenever some $w \Vdash \langle \sup \rangle \varphi_0 \varphi_0' \wedge \langle \sup \rangle \varphi_1 \varphi_1'$ in some semilattice model, the sub-semilattice generated by w and witnesses for $\{\langle \sup \rangle \varphi_0 \varphi_0', \langle \sup \rangle \varphi_1 \varphi_1'\}$ is

[1] I assume familiarity with such axiomatization practice, but, even so, I am aware that what follows may still require significant effort to fully understand. Nonetheless, I believe that my talk will provide further clarity. In this vein, let me finally stress that what follows is not clear-cut mathematical arguments, but heuristic guidance. It is not intended to be 'literally true' but 'metaphorically helpful'—hopefully not least for applying similar ideas and drawing inspiration in other axiomatization settings.

isomorphic to the RHS semilattice. But it is true that this sub-semilattice will be the (semilattice) *homorphic image* of the RHS semilattice. Moreover, this can adequately be encoded into the formula π_1 and will suffice for dealing with this first complication. This helps explain the following parts of the axiomatization:

- The axioms, like π_1, will be implications that can be intuited as follows: given the satisfaction of some formulas (the antecedent), a certain sub-semilattice is the homomorphic image of a certain other semilattice which is freely generated modulo some specified requirements (the consequent).

- To define formulas like π_1, we must, first, define this "certain other semilattice" which is "freely generated modulo specified requirements". This is formalized by taking freely generated semilattices $\mathcal{P}(S)\setminus\{\varnothing\}$ and quotienting out under the least congruence relation $\sim$ meeting the given requirements.

Continuing the stepwise construction, suppose, say, $\langle\sup\rangle\psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$. Again, simply adding corresponding worlds $\{\boldsymbol{\psi}\}, \{\boldsymbol{\psi'}\}$ labeled using the existence lemma for $l(\{\boldsymbol{\varphi_0}\})$ does not do the job. Because then, for instance, $\{*\} = \sup\{\{\boldsymbol{\psi'}\}, \{\boldsymbol{\varphi'_0}\}\}$ while we need not have $C_{\mathbf{Sem}}l(\{*\})l(\{\boldsymbol{\psi'}\})l(\{\boldsymbol{\varphi'_0}\})$. Once more, the solution must be to have some formula $\pi_2 \in \mathit{MIL}_{Sem}$ enabling us to construct an extended semilattice freely generated modulo the obvious requirements so that, crucially, $x = \sup\{y, z\}$ implies $C_{\mathbf{Sem}}l(x)l(y)l(z)$. This brings about a second (minor) complication: since $\langle\sup\rangle\psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$, it is instinctive to want to find a formula $\pi_2 \in \mathit{MIL}_{Sem}$ ascertaining this when 'evaluated at' $l(\{\boldsymbol{\varphi_0}\})$; however, $\mathcal{L}_M$-formulas can, clearly, only express properties of worlds below any given world of evaluation. Thus, there can be no formula π_2 expressing the desired when evaluated at $l(\{\boldsymbol{\varphi_0}\})$. Fortunately, a solution can be found: $l(\{*\}) = \Gamma_0$ is all-seeing (backwardly), so we should (and will) be able to express the desired with a formula π_2 evaluated at $l(\{*\}) = \Gamma_0$. Before going any further, let us summarize the key take-aways.

- To achieve the truth lemma, we will need to unboundedly extend the semilattice under construction. This explains one way in which our axiomatization is infinite: having, e.g., defined the RHS semilattice $\big(\mathcal{P}(S_1)\setminus\{\varnothing\}, \cup\big)/_{\sim_1}$ using the formula π_1, if, e.g., $\langle\sup\rangle\psi\psi' \in l(\{\boldsymbol{\varphi_0}\})$, we will need to construct an *extended* semilattice $\big(\mathcal{P}(S_2)\setminus\{\varnothing\}, \cup\big)/_{\sim_2}$ using a formula π_2. And then an extended one using a formula π_3, etc. That is, we must be able to ascertain that an ever-increasing sub-semilattice is the homormorphic image of a correspondingly ever-increasing semilattice freely generated modulo ever-more specified requirements.

- In a sense, the item above explains a way in which we must include axioms for each 'depth' $n \in \omega$. On top of that, we must also include axioms for each 'width' $n \in \omega$: the semilattice freely generated modulo requirements of $\{*\} = \sup\{\boldsymbol{\varphi_0}, \boldsymbol{\varphi'_0}\}$ and $\{*\} = \sup\{\boldsymbol{\varphi_1}, \boldsymbol{\varphi'_1}\}$ is obviously smaller than the one generated modulo requirements of $\{*\} = \sup\{\{\boldsymbol{\varphi_0}\}, \{\boldsymbol{\varphi'_0}\}\}, \{*\} = \sup\{\{\boldsymbol{\varphi_1}\}, \{\boldsymbol{\varphi'_1}\}\}$ and $\{*\} = \sup\{\{\boldsymbol{\varphi_2}\}, \{\boldsymbol{\varphi'_2}\}\}$, etc.

- When constructing the model to ensure that $x = \sup\{y, z\}$ always implies $C_{\mathbf{Sem}}l(x)l(y)l(z)$, we have to label all points with MCSs obtained by evaluating the formulas $\pi_1, \pi_2, \ldots$ at the top MCS $l(\{*\}) = \Gamma_0$.

Continuing, although solving one problem, this last solution of evaluating at $l(\{*\})$ inevitably constructs another (major) problem: having first labeled, e.g., $\{\boldsymbol{\varphi_0}\}$ via evaluating the formula π_1 at Γ_0, we now relabel $\{\boldsymbol{\varphi_0}\}$ via evaluating another formula π_2 at Γ_0. How then are we to ascertain that π_2 and π_1 agree on the labeling; i.e., that, e.g., $l_2(\{\boldsymbol{\varphi_0}\}) = l_1(\{\boldsymbol{\varphi_0}\})$? If we by using formulas somehow could 'name' the MCSs of the labeling induced by π_1, we could construct π_2 using these 'names' as to ensure that the labeling of $\{\boldsymbol{\varphi_0}\}$ induced by π_2 agrees with the labeling of π_1; thus, solving the problem. Evidently, (without nominals) there can be no way of doing so when dealing with MCSs. There is an alternative, though: while an MCS Θ is equivalently defined as an *infinite* conjunction $\widehat{\Theta}$, a finite set of formulas Θ_F is equivalently defined as a *finite* conjunction $\widehat{\Theta_F}$; i.e., in some sense, going finite facilitates 'naming'. This suggests the following idea:

- Instead of starting out with some (possibly infinite) consistent set X_0, we go for weak completeness and start with a consistent formula φ which we extend to the least subformula-closed set Φ containing $\{\varphi\}$. We then label our worlds according to which Φ-formulas they satisfy instead of with MCSs. In this way, using finite conjunctions, we can contain the labeling in the formula π_1, and then also in the extended formula π_2, etc. We then get that (1) $x = \sup\{y, z\}$ implies $C_{\mathbf{Sem}}\Gamma_x\Gamma_y\Gamma_z$ for some MCSs $\Gamma_i \supseteq l(i)$, and, importantly, (2) $l_1(x) = l_2(x)$.

Yet again, solving one problem we have caused another: how can π_1 also contain the information determining what Φ-formulas the worlds are to satisfy *and* still be valid: that, say, some $w \Vdash \langle \sup \rangle \varphi_0 \varphi_0'$ does not determine what Φ-formulas the witnessing φ_0- and φ_0'-worlds satisfy. Key here is that Φ is finite, so there are only finitely many 'names' over Φ, and we do know that the witnessing φ_0- and φ_0'-worlds must have some 'Φ-name'. Therefore, the consequent of π_1 will not state that one particular sub-semilattice is the homomorphic image of one particular other semilattice, but instead disjunctively quantify over all such options induced by all possible Φ-names. This brings us to our final point of elaboration:

- If the consequents of the formulas $\pi_1, \pi_2, \ldots$ consist of disjunctions defining *distinct* semilattices, which disjunct shall we choose when stepwise extending our semilattice as to satisfy the truth lemma? To answer this, it is helpful recalling how π_2 is to 'extend' π_1. Essentially, we want π_2 to encode how a bigger sub-semilattice must also be the homomorphic image of another bigger semilattice. So since each disjunct of the consequent of π_1 encodes how a sub-semilattice is the homomorphic image of another semilattice, π_2 must encode the extended claim for each disjunct. To do so, π_2 must, in particular, split each disjunct of π_1 into further disjunctions to quantify over all possible Φ-names for the 'new worlds' of the extended semilattices. And so forth as for $\pi_3, \ldots$.

  What we are left with is a tree where each node at each layer i, in particular, defines a semilattice (the one that a given sub-semilattice must be the homomorphic image of) and also a corresponding disjunct of a formula π_i, and the edges mark 'extension' of both semilattices and formulas. The main observations are then (1) the tree is finitely branching, and (2) we can assure that at each layer at least one disjunct must be 'satisfied', allowing for an infinite subtree where König's Lemma applies to supply an infinite chain of semilattice models of which its colimit is our satisfying semilattice model.

This concludes our 'study guide' for the axiomatization we are to present. While it is in no way exhaustive and should be treated as nothing but an informal heuristical guide, we hope that by having highlighted particular features, we have called to the fore ideas potentially applicable in other axiomatization settings.

References

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